Ember

EDUCATION

Vectors II

H2 MATHEMATICS · 9758

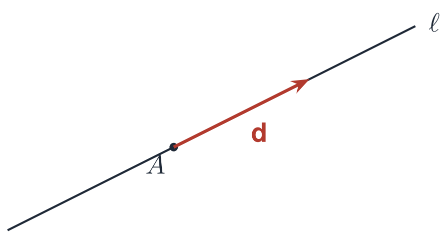
♦ EQUATIONS OF A LINE

Vector Equation

A line ℓ through a point A (position vector **a**) with **direction vector d** $\neq$ **0** has the vector equation

$$\ell : \mathbf{r} = \mathbf{a} + \lambda\mathbf{d}, \qquad \lambda \in \mathbb{R}.$$

- **a**: any fixed point on the line.
- **d**: any non-zero direction vector (replacing **d** by any non-zero scalar multiple gives the same line).
- $\lambda \in \mathbb{R}$: parameter; each value of λ gives one point on ℓ.



Cartesian Equation

Writing $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ and eliminating λ from the three component equations:

$$\frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}.$$

(Valid when all $d_i \neq 0$. If any $d_i = 0$, write that component as $x = a_1$ etc. separately.)

Point on a Line

To say "P is a point on ℓ" means $\overrightarrow{OP} = \mathbf{a} + \lambda\mathbf{d}$ **for some** $\lambda \in \mathbb{R}$. This phrasing matters in solving problems: we name P as a generic point with *one* unknown λ to be found later.

♦ Example 1.

The line ℓ passes through $A(2,5,1)$ and $B(-1,1,2)$.

(i) Find a vector equation of ℓ.
(ii) Determine whether $C(8,13,-1)$ lies on ℓ.

Solution.

(i) Direction vector:

$$\mathbf{d} = \overrightarrow{AB} = \begin{pmatrix}-1\\1\\2\end{pmatrix} - \begin{pmatrix}2\\5\\1\end{pmatrix} = \begin{pmatrix}-3\\-4\\1\end{pmatrix}.$$

Using A as the fixed point:

$$\ell : \mathbf{r} = \begin{pmatrix}2\\5\\1\end{pmatrix} + \lambda\begin{pmatrix}-3\\-4\\1\end{pmatrix}, \qquad \lambda \in \mathbb{R}.$$

(ii) C is on ℓ iff there exists $\lambda \in \mathbb{R}$ with $\overrightarrow{OC} = \mathbf{a} + \lambda\mathbf{d}$:

$$\begin{pmatrix}8\\13\\-1\end{pmatrix} = \begin{pmatrix}2\\5\\1\end{pmatrix} + \lambda\begin{pmatrix}-3\\-4\\1\end{pmatrix}.$$

Bringing $\begin{pmatrix}2\\5\\1\end{pmatrix}$ over to the left:

$$\begin{pmatrix}8\\13\\-1\end{pmatrix} - \begin{pmatrix}2\\5\\1\end{pmatrix} = \lambda\begin{pmatrix}-3\\-4\\1\end{pmatrix}.$$

Simplifying the left-hand side:

$$\begin{pmatrix}6\\8\\-2\end{pmatrix} = \lambda\begin{pmatrix}-3\\-4\\1\end{pmatrix}.$$

Factorising out -2 on the left:

$$-2\begin{pmatrix}-3\\-4\\1\end{pmatrix} = \lambda\begin{pmatrix}-3\\-4\\1\end{pmatrix}.$$

Both sides now contain the same direction vector, so $\lambda = -2 \in \mathbb{R}$. Therefore C *lies on* ℓ.

Example 2.

Write the line $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\lambda \in \mathbb{R}$, in Cartesian form.

Solution.

Equating components of $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$:

$$x = 2 + \lambda, \quad y = 0 + 2\lambda, \quad z = -3 - \lambda.$$

Solving each for λ:

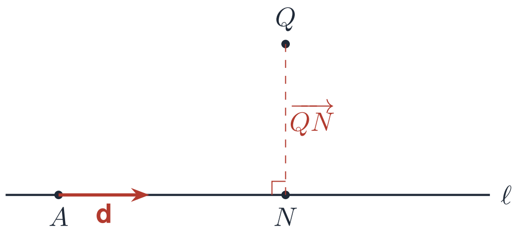
$$\lambda = x - 2 = \frac{y}{2} = \frac{z+3}{-1}.$$

Therefore

$$\frac{x-2}{1} = \frac{y}{2} = \frac{z+3}{-1}.$$

From a Point to a Line

Given a point Q and a line $\ell : \mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, $\lambda \in \mathbb{R}$, the **foot of perpendicular** N from Q to ℓ is the point on ℓ where $\overrightarrow{QN} \perp \mathbf{d}$.



METHOD (foot of perpendicular to a line)

1. Let N be on ℓ, so $\overrightarrow{ON} = \mathbf{a} + \mu\mathbf{d}$ for some $\mu \in \mathbb{R}$.
2. Form $\overrightarrow{QN} = \overrightarrow{ON} - \overrightarrow{OQ}$, still in terms of μ.
3. Apply $\overrightarrow{QN} \cdot \mathbf{d} = 0$ and solve for μ.
4. Substitute μ back to get $\overrightarrow{ON}$. The shortest distance is $\left|\overrightarrow{QN}\right|$.

Shortest Distance: Two Methods

(A) Via foot of perpendicular: compute N as above, then $|QN|$.

(B) Direct projection formula: $\text{dist}(Q, \ell) = \dfrac{\left|\overrightarrow{AQ} \times \mathbf{d}\right|}{|\mathbf{d}|}$ where A is any point on ℓ.

♦ Example 3.

Find the foot of perpendicular N from $Q(4,1,0)$ to the line $\ell:$ $\mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-1\end{pmatrix}$, $\lambda \in \mathbb{R}$. Hence find the shortest distance from Q to ℓ.

Solution.

Let N be on ℓ, so $\overrightarrow{ON} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \mu\begin{pmatrix}1\\2\\-1\end{pmatrix}$ for some $\mu \in \mathbb{R}$.

Compute $\overrightarrow{QN}$:

$$\overrightarrow{QN} = \overrightarrow{ON} - \overrightarrow{OQ} = \begin{pmatrix}1+\mu-4\\2\mu-1\\2-\mu-0\end{pmatrix} = \begin{pmatrix}\mu-3\\2\mu-1\\2-\mu\end{pmatrix}.$$

Apply $\overrightarrow{QN}\cdot\mathbf{d} = 0$:

$$\begin{pmatrix}\mu-3\\2\mu-1\\2-\mu\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix} = 0.$$

$$(\mu-3)(1) + (2\mu-1)(2) + (2-\mu)(-1) = 0.$$

$$\mu - 3 + 4\mu - 2 - 2 + \mu = 0.$$

$$6\mu - 7 = 0 \implies \mu = \tfrac{7}{6}.$$

Substitute back:

$$\overrightarrow{ON} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \tfrac{7}{6}\begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}13/6\\7/3\\5/6\end{pmatrix}, \quad \text{so } N = \left(\tfrac{13}{6}, \tfrac{7}{3}, \tfrac{5}{6}\right).$$

Shortest distance:

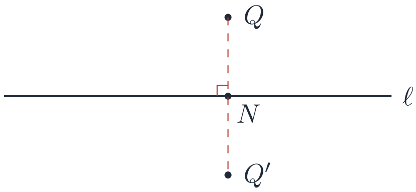
$$\overrightarrow{QN} = \begin{pmatrix}\frac{7}{6}-3\\\frac{7}{3}-1\\\frac{5}{6}-0\end{pmatrix} = \begin{pmatrix}-11/6\\4/3\\5/6\end{pmatrix}.$$

$$\left|\overrightarrow{QN}\right| = \sqrt{\left(\tfrac{11}{6}\right)^2 + \left(\tfrac{4}{3}\right)^2 + \left(\tfrac{5}{6}\right)^2} = \sqrt{\tfrac{121+64+25}{36}} = \sqrt{\tfrac{210}{36}} = \tfrac{\sqrt{210}}{6}.$$

Reflection of a Point in a Line

The **reflection** of a point Q in a line ℓ is the point Q' such that ℓ is the perpendicular bisector of QQ'. Since the foot of perpendicular N from Q to ℓ is the midpoint of QQ':

> ♦ **METHOD (reflection of a point in a line)**
> 1. Find the foot of perpendicular N from Q to ℓ (as in the method above).
> 2. Use the mid-point relation: $\overrightarrow{OQ'} = 2\,\overrightarrow{ON} - \overrightarrow{OQ}$.



♦ **Example 4.**

Find the reflection Q' of $Q(3,1,0)$ in the line $\ell:\ \mathbf{r} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\lambda \in \mathbb{R}$.

Solution.

Let N be on ℓ, so $\overrightarrow{ON} = \begin{pmatrix}1+\mu\\ \mu\\ \mu\end{pmatrix}$ for some $\mu \in \mathbb{R}$.

Compute $\overrightarrow{QN}$:

$$\overrightarrow{QN} = \overrightarrow{ON} - \overrightarrow{OQ} = \begin{pmatrix}\mu-2\\ \mu-1\\ \mu\end{pmatrix}.$$

Apply $\overrightarrow{QN}\cdot\mathbf{d} = 0$ where $\mathbf{d} = \begin{pmatrix}1\\1\\1\end{pmatrix}$:

$$\begin{pmatrix}\mu-2\\ \mu-1\\ \mu\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} = 0 \implies 3\mu - 3 = 0 \implies \mu = 1.$$

So $N = (2,1,1)$.

Reflection:

$$\overrightarrow{OQ'} = 2\,\overrightarrow{ON} - \overrightarrow{OQ} = 2\begin{pmatrix}2\\1\\1\end{pmatrix} - \begin{pmatrix}3\\1\\0\end{pmatrix} = \begin{pmatrix}4\\2\\2\end{pmatrix} - \begin{pmatrix}3\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix}.$$

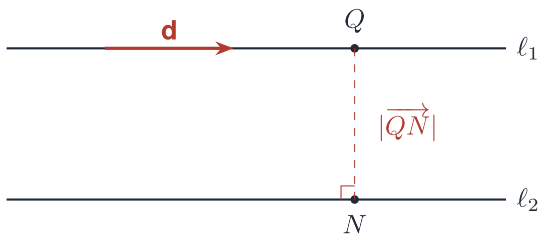
The reflection is $Q' = (1,1,2)$.

Distance Between Two Parallel Lines

Two lines are **parallel** if their direction vectors are parallel (i.e. one is a scalar multiple of the other). The shortest distance is found using the foot of perpendicular:

♦ **METHOD (distance between parallel lines)**
Given $\ell_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ and $\ell_2 : \mathbf{r} = \mathbf{c} + \mu\mathbf{d}$:

1. Let $Q = A$ (any point on ℓ_1). Let N be on ℓ_2: $\overrightarrow{ON} = \mathbf{c} + t\mathbf{d}$ for some $t \in \mathbb{R}$.
2. Form $\overrightarrow{QN} = \overrightarrow{ON} - \overrightarrow{OQ}$, still in terms of t.
3. Apply $\overrightarrow{QN} \cdot \mathbf{d} = 0$ and solve for t.
4. Shortest distance $= |\overrightarrow{QN}|$.



♦ **Example 5.**

The lines

$$\ell_1 : \mathbf{r} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-2\end{pmatrix}, \qquad \ell_2 : \mathbf{r} = \begin{pmatrix}2\\3\\1\end{pmatrix} + \mu\begin{pmatrix}2\\1\\-2\end{pmatrix}, \qquad \lambda, \mu \in \mathbb{R},$$

are parallel. Find the shortest distance between ℓ_1 and ℓ_2.

Solution.

Let $Q = A = (1, 0, 0)$ on ℓ_1. Let N be on ℓ_2: $\overrightarrow{ON} = \begin{pmatrix}2+2t\\3+t\\1-2t\end{pmatrix}$ for some $t \in \mathbb{R}$.

Compute $\overrightarrow{QN}$:

$$\overrightarrow{QN} = \overrightarrow{ON} - \overrightarrow{OQ} = \begin{pmatrix}1+2t\\3+t\\1-2t\end{pmatrix}.$$

Apply $\overrightarrow{QN} \cdot \mathbf{d} = 0$ where $\mathbf{d} = \begin{pmatrix}2\\1\\-2\end{pmatrix}$:

$$\begin{pmatrix}1+2t\\3+t\\1-2t\end{pmatrix} \cdot \begin{pmatrix}2\\1\\-2\end{pmatrix} = 0.$$

$$2(1+2t) + (3+t) + (-2)(1-2t) = 0.$$

$$3 + 9t = 0 \implies t = -\tfrac{1}{3}.$$

Substitute back:

$$\overrightarrow{ON} = \begin{pmatrix} 4/3 \\ 8/3 \\ 5/3 \end{pmatrix}, \qquad \overrightarrow{QN} = \begin{pmatrix} 1/3 \\ 8/3 \\ 5/3 \end{pmatrix}.$$

$$\left|\overrightarrow{QN}\right| = \sqrt{\tfrac{1}{9} + \tfrac{64}{9} + \tfrac{25}{9}} = \sqrt{\tfrac{90}{9}} = \sqrt{10}.$$

Two Lines

For two lines $\ell_1 : \mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and $\ell_2 : \mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$ (with $\lambda, \mu \in \mathbb{R}$), there are exactly four possibilities:

- **Parallel and distinct:** $\mathbf{d}_1 \parallel \mathbf{d}_2$ but $\mathbf{a}_2 - \mathbf{a}_1 \nparallel \mathbf{d}_1$. No intersection.
- **Identical:** $\mathbf{d}_1 \parallel \mathbf{d}_2$ and $\mathbf{a}_2 - \mathbf{a}_1 \parallel \mathbf{d}_1$. Same line.
- **Intersecting:** $\mathbf{d}_1 \nparallel \mathbf{d}_2$ and the parametric equations have a common solution (λ, μ). One intersection point.
- **Skew:** $\mathbf{d}_1 \nparallel \mathbf{d}_2$ *and* no common solution. Lines do not meet and are not parallel.

♦ **METHOD (test intersect vs. skew, given $\mathbf{d}_1 \nparallel \mathbf{d}_2$)**

1. Set $\mathbf{a}_1 + \lambda\mathbf{d}_1 = \mathbf{a}_2 + \mu\mathbf{d}_2$, giving three component equations in λ, μ.
2. Solve any two for λ and μ.
3. Substitute into the third equation: if it holds, lines intersect; substitute back to get the point. If not, lines are skew.

♦ Example 6.

The lines ℓ_1 and ℓ_2 have vector equations

$$\ell_1 : \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \ell_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}.$$

Show that ℓ_1 and ℓ_2 intersect, and find the coordinates of the point of intersection.

Solution.

Direction vectors are not parallel ($\mathbf{d}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is not a scalar multiple of $\mathbf{d}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$), so ℓ_1 and ℓ_2 are either intersecting or skew.

Set $\mathbf{a}_1 + \lambda\mathbf{d}_1 = \mathbf{a}_2 + \mu\mathbf{d}_2$:

$$\begin{pmatrix} 1 + \lambda \\ -1 - \lambda \\ 3 + \lambda \end{pmatrix} = \begin{pmatrix} 2 + 2\mu \\ 4 + \mu \\ 6 + 3\mu \end{pmatrix}.$$

From the x-component: $1 + \lambda = 2 + 2\mu$, so $\lambda - 2\mu = 1$. (1)
From the y-component: $-1 - \lambda = 4 + \mu$, so $\lambda + \mu = -5$. (2)

Subtract (1) from (2): $3\mu = -6$, so $\mu = -2$ and $\lambda = -3$.

Check the z-component: $3 + (-3) = 0$ and $6 + 3(-2) = 0$. ✓

The lines *intersect*. Substitute $\lambda = -3$:

$$\overrightarrow{OE} = \begin{pmatrix} 1-3 \\ -1+3 \\ 3-3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}, \quad E = (-2, 2, 0).$$

♦ Example 7.

The lines ℓ_1 and ℓ_2 have vector equations

$$\ell_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \quad \ell_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}.$$

Show that ℓ_1 and ℓ_2 are skew.

Solution.

Direction vectors: $\mathbf{d}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{d}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

These are not parallel (no scalar k with $\mathbf{d}_1 = k\mathbf{d}_2$). So ℓ_1 and ℓ_2 are either intersecting or skew.

Set $\mathbf{a}_1 + \lambda\mathbf{d}_1 = \mathbf{a}_2 + \mu\mathbf{d}_2$:

$$\begin{pmatrix} 1 + 2\lambda \\ \lambda \\ 2 - \lambda \end{pmatrix} = \begin{pmatrix} 3 + \mu \\ 2 - \mu \\ 2\mu \end{pmatrix}.$$

Equating the x-components:

$$1 + 2\lambda = 3 + \mu \implies 2\lambda - \mu = 2. \qquad (1)$$

Equating the y-components:

$$\lambda = 2 - \mu \implies \lambda + \mu = 2. \qquad (2)$$

Adding (1) and (2):

$$3\lambda = 4 \implies \lambda = \tfrac{4}{3}, \quad \mu = \tfrac{2}{3}.$$

Now check the z-components:

$$\text{LHS} = 2 - \tfrac{4}{3} = \tfrac{2}{3}, \qquad \text{RHS} = 2 \cdot \tfrac{2}{3} = \tfrac{4}{3}.$$

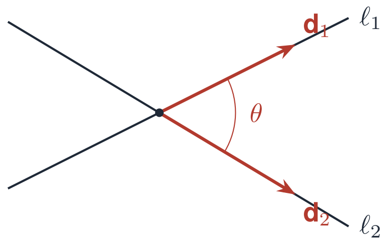
Since LHS $\neq$ RHS, the three component equations have no common solution. The directions are not parallel either, so ℓ_1 and ℓ_2 are *skew*.

Angle Between Two Lines

Acute angle θ between lines with direction vectors $\mathbf{d}_1, \mathbf{d}_2$:

$$\cos\theta = \frac{|\mathbf{d}_1 \cdot \mathbf{d}_2|}{|\mathbf{d}_1|\ |\mathbf{d}_2|}.$$

The absolute value ensures we always get the acute angle: if the directions point the same general way, $\mathbf{d}_1 \cdot \mathbf{d}_2 > 0$ gives the angle directly; if they point opposite ways, $|\mathbf{d}_1 \cdot \mathbf{d}_2|$ flips it to the acute side.



♦ Example 8.

Find the acute angle between the lines with direction vectors $\mathbf{d}_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $\mathbf{d}_2 = \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$.

Solution.

Magnitudes:

$$|\mathbf{d}_1| = \sqrt{1+4+4} = 3, \quad |\mathbf{d}_2| = \sqrt{9+16+0} = 5.$$

Dot product:

$$\mathbf{d}_1 \cdot \mathbf{d}_2 = (1)(3) + (2)(-4) + (-2)(0) = -5.$$

$$\cos\theta = \frac{|-5|}{3 \cdot 5} = \frac{1}{3}.$$

$$\theta = \cos^{-1}\left(\tfrac{1}{3}\right) \approx 70.5^\circ.$$

EQUATIONS OF A PLANE

A plane π in 3-D can be specified in three equivalent forms.

Vector (Parametric) Form

Through point A (position $\mathbf{a}$) with two non-parallel direction vectors $\mathbf{d}_1, \mathbf{d}_2$ lying in the plane:

$$\pi : \mathbf{r} \; = \; \mathbf{a} + \lambda\mathbf{d}_1 + \mu\mathbf{d}_2, \qquad \lambda, \mu \in \mathbb{R}.$$

Scalar Product (Normal) Form

Through A with normal vector $\mathbf{n}$ (perpendicular to every direction in the plane):

$$\pi : \mathbf{r}\cdot\mathbf{n} \; = \; \mathbf{a}\cdot\mathbf{n} \; = \; D.$$

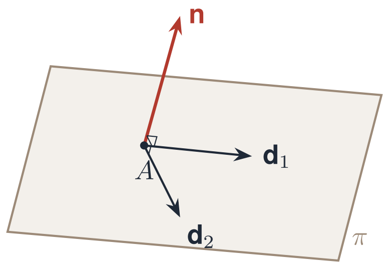
Cartesian Form

Writing $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$:

$$\pi : \; n_1x + n_2y + n_3z \; = \; D.$$

Finding the Normal from Vector Form

If you are given $\mathbf{d}_1, \mathbf{d}_2$, take $\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2$. Then $D = \mathbf{a}\cdot\mathbf{n}$.



Example 9.

Find a scalar-product equation of the plane π passing through the points $A(1, 0, 2)$, $B(3, 1, 0)$ and $C(0, 2, 1)$.

Solution.

Two directions in π:

$$\overrightarrow{AB} \; = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \qquad \overrightarrow{AC} \; = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$

Normal vector:

$$\mathbf{n} \; = \; \overrightarrow{AB} \times \overrightarrow{AC} \; = \begin{pmatrix} (1)(-1) - (-2)(2) \\ (-2)(-1) - (2)(-1) \\ (2)(2) - (1)(-1) \end{pmatrix} \; = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}.$$

Right-hand side:

$$D = \mathbf{a}\cdot\mathbf{n} = \begin{pmatrix}1\\0\\2\end{pmatrix}\cdot\begin{pmatrix}3\\4\\5\end{pmatrix} = (1)(3)+(0)(4)+(2)(5) = 13.$$

Therefore

$$\pi : \mathbf{r}\cdot\begin{pmatrix}3\\4\\5\end{pmatrix} = 13, \quad \text{i.e. } 3x+4y+5z=13.$$

Example 10.

Convert the plane $\mathbf{r} = \begin{pmatrix}1\\0\\-1\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\3\end{pmatrix} + \mu\begin{pmatrix}-1\\2\\0\end{pmatrix}$, $\lambda,\mu\in\mathbb{R}$, to scalar-product form.

Solution.

Normal vector:

$$\mathbf{n} = \begin{pmatrix}2\\1\\3\end{pmatrix}\times\begin{pmatrix}-1\\2\\0\end{pmatrix} = \begin{pmatrix}(1)(0)-(3)(2)\\(3)(-1)-(2)(0)\\(2)(2)-(1)(-1)\end{pmatrix} = \begin{pmatrix}-6\\-3\\5\end{pmatrix}.$$

Right-hand side, using $\mathbf{a} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$:

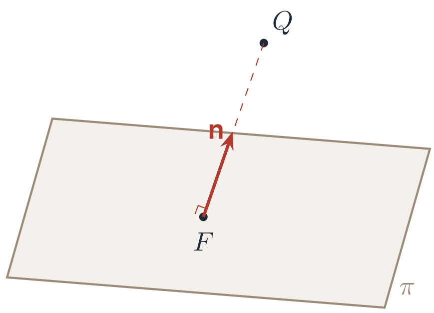
$$D = \mathbf{a}\cdot\mathbf{n} = (1)(-6)+(0)(-3)+(-1)(5) = -11.$$

Therefore

$$\pi : \mathbf{r}\cdot\begin{pmatrix}-6\\-3\\5\end{pmatrix} = -11, \quad \text{i.e. } -6x-3y+5z=-11.$$

From a Point to a Plane

Given Q and plane $\pi:\ \mathbf{r}\cdot\mathbf{n}=D$, the **foot of perpendicular** F from Q to π is the point on π where $\overrightarrow{QF}$ is parallel to **n** (because $\overrightarrow{QF}\perp$ plane $\Leftrightarrow \overrightarrow{QF}\parallel \mathbf{n}$).



♦ **METHOD (foot of perpendicular to a plane)**

1. The line through Q perpendicular to π is $\mathbf{r}=\overrightarrow{OQ}+t\mathbf{n}$, $t\in\mathbb{R}$.
2. F lies on this line *and* on π: substitute into $\mathbf{r}\cdot\mathbf{n}=D$ to solve for t.
3. Substitute t back to get $\overrightarrow{OF}$. Shortest distance is $\left|\overrightarrow{QF}\right|=|t|\ |\mathbf{n}|$.

Shortest Distance

Direct formula:

$$\text{dist}(Q,\pi)\ =\ \frac{|\,\overrightarrow{OQ}\cdot\mathbf{n}-D\,|}{|\mathbf{n}|}.$$

♦ Example 11.

Find the foot of perpendicular F from $Q(3,1,-2)$ to the plane $\pi:\ x+2y-z=4$. Hence find the shortest distance from Q to π.

Solution.

Normal: $\mathbf{n}=\begin{pmatrix}1\\2\\-1\end{pmatrix}$, $D=4$.

Line through Q perpendicular to π:

$$\mathbf{r}\ =\ \begin{pmatrix}3\\1\\-2\end{pmatrix}+t\begin{pmatrix}1\\2\\-1\end{pmatrix},\quad t\in\mathbb{R}.$$

For F on this line and on π, $\mathbf{r}\cdot\mathbf{n}=4$:

$$\left(\begin{pmatrix}3\\1\\-2\end{pmatrix}+t\begin{pmatrix}1\\2\\-1\end{pmatrix}\right)\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix}\ =\ 4.$$

$$\big[(3)(1)+(1)(2)+(-2)(-1)\big]+t\big[(1)(1)+(2)(2)+(-1)(-1)\big]\ =\ 4.$$

$$7 + 6t = 4 \implies t = -\tfrac{1}{2}.$$

Substitute back:

$$\overrightarrow{OF} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 0 \\ -3/2 \end{pmatrix}, \quad F = \left(\tfrac{5}{2}, 0, -\tfrac{3}{2}\right).$$

Shortest distance: first find $\overrightarrow{QF}$,

$$\overrightarrow{QF} = \overrightarrow{OF} - \overrightarrow{OQ} = \begin{pmatrix} 5/2 \\ 0 \\ -3/2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1 \\ 1/2 \end{pmatrix}.$$

$$\left|\overrightarrow{QF}\right| = \sqrt{\tfrac{1}{4} + 1 + \tfrac{1}{4}} = \sqrt{\tfrac{6}{4}} = \tfrac{\sqrt{6}}{2}.$$

♦ Example 12.

Find the distance from $Q(2, -1, 3)$ to the plane $\pi : 2x - y + 2z = 5$ using *both* the foot-of-perpendicular method and the direct formula.

Solution.

(A) Foot of perpendicular. Normal $\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, $|\mathbf{n}| = \sqrt{4 + 1 + 4} = 3$.

Line through Q along $\mathbf{n}$: $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, $t \in \mathbb{R}$.

Substitute into $\mathbf{r} \cdot \mathbf{n} = 5$:

$$\big[(2)(2) + (-1)(-1) + (3)(2)\big] + t\big[4 + 1 + 4\big] = 5.$$

$$11 + 9t = 5 \implies t = -\tfrac{2}{3}.$$

Foot of perpendicular:

$$\overrightarrow{OF} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - \tfrac{2}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -1/3 \\ 5/3 \end{pmatrix}.$$

Then $\overrightarrow{QF} = \overrightarrow{OF} - \overrightarrow{OQ}$ and find its length:

$$\overrightarrow{QF} = \begin{pmatrix} 2/3 \\ -1/3 \\ 5/3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -4/3 \\ 2/3 \\ -4/3 \end{pmatrix}.$$

$$\left|\overrightarrow{QF}\right| = \sqrt{\tfrac{16}{9} + \tfrac{4}{9} + \tfrac{16}{9}} = \sqrt{\tfrac{36}{9}} = 2.$$

(B) Direct formula.

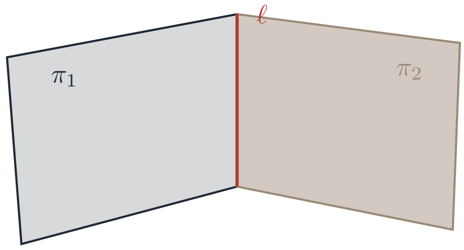
$$\operatorname{dist}(Q, \pi) = \frac{|\,\overrightarrow{OQ} \cdot \mathbf{n} - D\,|}{|\mathbf{n}|} = \frac{|11 - 5|}{3} = \frac{6}{3} = 2. \quad \checkmark$$

Both methods give the same distance 2.

Two Planes

Two distinct planes π_1 and π_2 with normals $\mathbf{n}_1, \mathbf{n}_2$:

- $\mathbf{n}_1 \parallel \mathbf{n}_2$**:** the planes are parallel (no intersection if distinct).
- $\mathbf{n}_1 \nparallel \mathbf{n}_2$**:** the planes meet in a *line* ℓ with direction $\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2$.



Line of Intersection of Two Planes

To find ℓ: (i) direction $\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2$; (ii) any point on both planes (set $z = 0$ say, solve for x, y).

♦ Example 13.

Find a vector equation of the line of intersection of the planes

$$\pi_1 : \ 2x + y - z = 3, \qquad \pi_2 : \ x - y + 2z = 1.$$

Solution.

$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{n}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. These are not parallel, so the planes meet in a line.

Direction:

$$\mathbf{d} \ = \ \mathbf{n}_1 \times \mathbf{n}_2 \ = \begin{pmatrix} (1)(2) - (-1)(-1) \\ (-1)(1) - (2)(2) \\ (2)(-1) - (1)(1) \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}.$$

Point on both planes: set $z = 0$:

$$2x + y = 3, \qquad x - y = 1.$$

Adding: $3x = 4$, so $x = \frac{4}{3}$, $y = \frac{1}{3}$.

Hence $A = \left(\frac{4}{3}, \frac{1}{3}, 0\right)$ lies on ℓ.

$$\ell : \ \mathbf{r} \ = \begin{pmatrix} 4/3 \\ 1/3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Using a GC to Find the Whole Line

The GC can return the entire line in one step. Treat the two Cartesian plane equations

$$\pi_1 : 2x + y - z = 3, \qquad \pi_2 : x - y + 2z = 1$$

as a system of 2 equations in 3 unknowns.

On the GC (TI-84): **APPS** $\to$ **PlySmlt2** $\to$ **SIMULTANEOUS EQN SOLVER**. Set **Equations** $= 2$, **Unknowns** $= 3$, then enter the coefficients (last column $=$ constants):

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 1 \end{array}\right].$$

Press **SOLVE**. The planes meet in a line, so there are infinitely many solutions and the GC writes everything in terms of the free variable $x_3 = z$:

$$x_1 = \tfrac{4}{3} - \tfrac{1}{3}x_3, \quad x_2 = \tfrac{1}{3} + \tfrac{5}{3}x_3, \quad x_3 = x_3.$$

Writing $x = x_1$, $y = x_2$ and $z = x_3 = t$:

$$\mathbf{r} = \begin{pmatrix} \frac{4}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} + t\begin{pmatrix} -\frac{1}{3} \\ \frac{5}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}, \quad \lambda \in \mathbb{R},$$

scaling the direction by -3 to clear fractions. This matches the cross-product direction $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}$ found above.

Angle Between Two Planes

Acute angle θ between planes with normals $\mathbf{n}_1, \mathbf{n}_2$:

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|}.$$

♦ Example 14.

Find the acute angle between the planes $\pi_1 :\ 3x - y + z = 2$ and $\pi_2 :\ x + y + 2z = 5$.

Solution.

$\mathbf{n}_1 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$, $|\mathbf{n}_1| = \sqrt{9+1+1} = \sqrt{11}$.

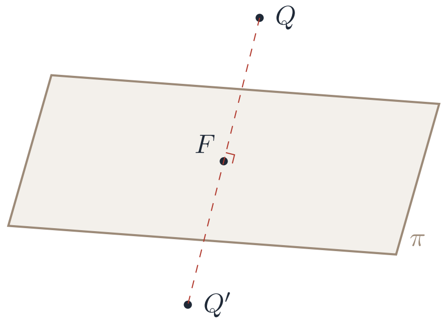
$\mathbf{n}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $|\mathbf{n}_2| = \sqrt{1+1+4} = \sqrt{6}$.

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = (3)(1) + (-1)(1) + (1)(2) = 4.$$

$$\cos\theta = \frac{|4|}{\sqrt{11}\cdot\sqrt{6}} = \frac{4}{\sqrt{66}}.$$

$$\theta = \cos^{-1}\left(\frac{4}{\sqrt{66}}\right) \approx 60.5°.$$

Reflection of a Point in a Plane



♦ METHOD (reflection in a plane)

1. Find the foot of perpendicular F from Q to π (as in Example 10).
2. Use the mid-point relation: $\overrightarrow{OF} = \frac{1}{2}(\overrightarrow{OQ} + \overrightarrow{OQ'})$, so $\overrightarrow{OQ'} = 2\overrightarrow{OF} - \overrightarrow{OQ}$.

♦ Example 15.

Find the reflection of $Q(3, 1, -2)$ in the plane $\pi : \ x + 2y - z = 4$.

Solution.

From Example 10, the foot of perpendicular is $F = (\frac{5}{2}, 0, -\frac{3}{2})$.

Mid-point relation:

$$\overrightarrow{OQ'} = 2\overrightarrow{OF} - \overrightarrow{OQ} = 2\begin{pmatrix} 5/2 \\ 0 \\ -3/2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}.$$

$$\overrightarrow{OQ'} = \begin{pmatrix} 5 \\ 0 \\ -3 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}.$$

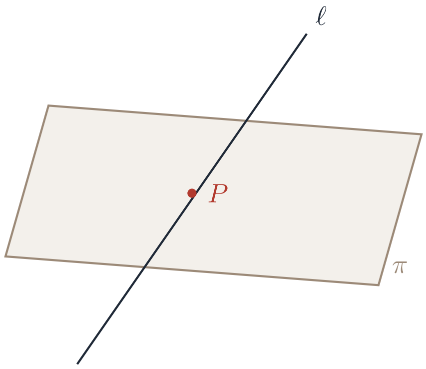
The reflection is $Q' = (2, -1, -1)$.

LINES AND PLANES

Line and Plane

For a line $\ell: \ \mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, $\lambda \in \mathbb{R}$, and plane $\pi: \ \mathbf{r} \cdot \mathbf{n} = D$, there are exactly three possibilities:

- $\mathbf{d} \cdot \mathbf{n} \neq 0$: ℓ pierces π at exactly one point.
- $\mathbf{d} \cdot \mathbf{n} = 0$ **and** $\mathbf{a} \cdot \mathbf{n} \neq D$: ℓ is parallel to π (no intersection).
- $\mathbf{d} \cdot \mathbf{n} = 0$ **and** $\mathbf{a} \cdot \mathbf{n} = D$: ℓ lies in π.



Finding the Point of Intersection

Substitute the parametric line into the plane equation: solve $(\mathbf{a} + \lambda\mathbf{d}) \cdot \mathbf{n} = D$ for λ, then substitute back.

Example 16.

The line $\ell: \ \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, $\lambda \in \mathbb{R}$, meets the plane $\pi: \ x + y + z = 5$. Find the point of intersection.

Solution.

Substitute $\mathbf{r}$ into $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 5$:

$$\left(\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 5.$$

Expanding:

$$\big[(1)(1) + (2)(1) + (-1)(1)\big] + \lambda\big[(2)(1) + (-1)(1) + (3)(1)\big] = 5.$$
$$2 + 4\lambda = 5 \implies \lambda = \tfrac{3}{4}.$$

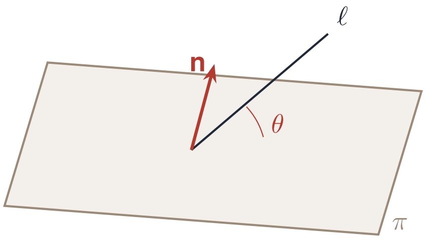
Substitute back:

$$\overrightarrow{OP} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \tfrac{3}{4} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 5/4 \\ 5/4 \end{pmatrix}, \quad P = \left(\tfrac{5}{2}, \tfrac{5}{4}, \tfrac{5}{4}\right).$$

Angle Between a Line and a Plane

Acute angle θ between line with direction **d** and plane with normal **n**:

$$\sin\theta = \frac{|\mathbf{d}\cdot\mathbf{n}|}{|\mathbf{d}|\,|\mathbf{n}|}.$$



♦ Example 17.

Find the acute angle between the line $\ell: \mathbf{r} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\lambda \in \mathbb{R}$, and the plane $\pi: x + 2y - z = 3$.

Solution.

Direction $\mathbf{d} = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $|\mathbf{d}| = \sqrt{3}$. Normal $\mathbf{n} = \begin{pmatrix}1\\2\\-1\end{pmatrix}$, $|\mathbf{n}| = \sqrt{6}$.

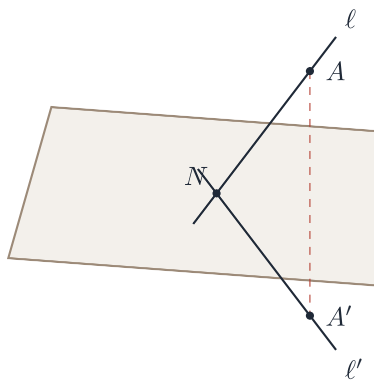
$$\mathbf{d}\cdot\mathbf{n} = (1)(1) + (1)(2) + (1)(-1) = 2.$$

$$\sin\theta = \frac{|2|}{\sqrt{3}\cdot\sqrt{6}} = \frac{2}{\sqrt{18}} = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3}.$$

$$\theta = \sin^{-1}\left(\frac{\sqrt{2}}{3}\right) \approx 28.1°.$$

Reflection of a Line in a Plane

To reflect a line ℓ in a plane p: if ℓ meets p, the reflected line passes through that intersection. Reflect any other point on ℓ to get a second point on the reflected line.



> ♦ **METHOD (reflection of a line in a plane)**
> 1. Find N: the point where ℓ meets p (substitute ℓ into p).
> 2. Choose any other point A on ℓ. Find its reflection A' in p (foot-of-perp method).
> 3. Reflected line ℓ': passes through N with direction $\overrightarrow{NA'}$.

♦ **Example 18.**

Line $\ell : \mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$, and plane $p : 2x + y + z = 6$. Find the vector equation of the reflection of ℓ in p.

Solution.

Step 1: Intersection N of ℓ with p.

Substituting ℓ into p:

$$2(3 + \lambda) + (1 - \lambda) + (2 + 2\lambda) = 6 \implies 9 + 3\lambda = 6 \implies \lambda = -1.$$

So $N = (2, 2, 0)$.

Step 2: Reflect $A = (3, 1, 2)$ (point on ℓ at $\lambda = 0$) in p.

The line through A perpendicular to p has normal $\mathbf{n} = (2, 1, 1)$: $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

Substituting into p: $2(3 + 2\mu) + (1 + \mu) + (2 + \mu) = 6 \implies 9 + 6\mu = 6 \implies \mu = -\frac{1}{2}$.

Foot $F = \left(2,\ \frac{1}{2},\ \frac{3}{2}\right)$.

$$\overrightarrow{OA'} = 2\,\overrightarrow{OF} - \overrightarrow{OA} = 2\begin{pmatrix} 2 \\ 1/2 \\ 3/2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

So $A' = (1, 0, 1)$.

Step 3: Reflected line ℓ'.

$$\overrightarrow{NA'} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}.$$

$$\boxed{\ell' : \mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \nu \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, \quad \nu \in \mathbb{R}.}$$

Points on a Line Equidistant from Two Planes

A point C is equidistant from planes p_1 and p_2 when $d(C, p_1) = d(C, p_2)$. Applying the point-to-plane distance formula and equating gives two cases.

♦ Example 19.

Line $m : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$. Planes $p_1 : x + 2y - 2z = 5$ and $p_2 : 2x - y + 2z = 1$. Find the coordinates of all points on m equidistant from p_1 and p_2.

Solution.

A general point on m is $C = (1 + t,\ 2 - t,\ -1 + t)$.

Since $|\mathbf{n}_1| = \sqrt{1 + 4 + 4} = 3$ and $|\mathbf{n}_2| = \sqrt{4 + 1 + 4} = 3$:

$$d(C, p_1) = \frac{|(1 + t) + 2(2 - t) - 2(-1 + t) - 5|}{3} = \frac{|2 - 3t|}{3},$$
$$d(C, p_2) = \frac{|2(1 + t) - (2 - t) + 2(-1 + t) - 1|}{3} = \frac{|5t - 3|}{3}.$$

Setting $d(C, p_1) = d(C, p_2)$: $|2 - 3t| = |5t - 3|$.

Case 1: $2 - 3t = 5t - 3 \implies 8t = 5 \implies t = \frac{5}{8}$. Point: $\left(\frac{13}{8}, \frac{11}{8}, -\frac{3}{8}\right)$.

Case 2: $2 - 3t = -(5t - 3) \implies 2t = 1 \implies t = \frac{1}{2}$. Point: $\left(\frac{3}{2}, \frac{3}{2}, -\frac{1}{2}\right)$.

APPENDIX: USEFUL DIRECT FORMULAS

1. Distance from Point Q to Plane $\pi : \mathbf{r} \cdot \mathbf{n} = D$

$$\text{dist} = \frac{|\overrightarrow{OQ} \cdot \mathbf{n} - D|}{|\mathbf{n}|}.$$

2. Distance from Point Q to Line $\ell : \mathbf{r} = \mathbf{a} + \lambda \mathbf{d}$

$$\text{dist} = \frac{\left|\overrightarrow{AQ} \times \mathbf{d}\right|}{|\mathbf{d}|}, \quad \text{where } A \text{ is any point on } \ell.$$

3. Distance Between Two Parallel Planes

For $\pi_1 : \mathbf{r} \cdot \mathbf{n} = D_1$ and $\pi_2 : \mathbf{r} \cdot \mathbf{n} = D_2$ (same $\mathbf{n}$): pick any point on π_1 and apply formula 1 to π_2 (equivalent to:)

$$\text{dist} = \frac{|D_1 - D_2|}{|\mathbf{n}|}.$$

4. Distance Between Two Parallel Lines

Given $\ell_1 : \mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ and $\ell_2 : \mathbf{r} = \mathbf{c} + \mu \mathbf{b}$ (parallel, same $\mathbf{b}$), with $A \in \ell_1$, $B \in \ell_2$:

$$\text{dist} = \frac{|\overrightarrow{AB} \times \mathbf{b}|}{|\mathbf{b}|}.$$