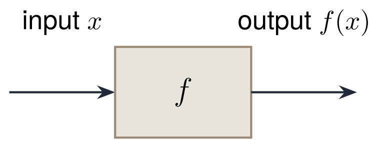
Ember
EDUCATION

Functions

H2 MATHEMATICS · 9758

WHAT IS A FUNCTION?

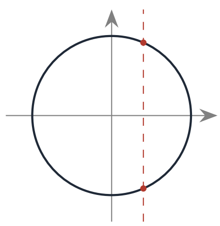
A **function** is a rule that takes an input and returns exactly one output. A function behaves like a machine: one input enters, one output leaves.



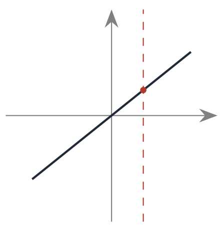
We write a function as $f : x \mapsto y$, meaning the rule f sends input x to output y. The set of allowed inputs is the **domain** D_f; the set of outputs the function actually produces is the **range** R_f.

The Vertical Line Test

A curve is the graph of a function only when each x gives exactly one y. Hence a vertical line meets the curve **at most once**.



Not a function (circle)



Is a function (line)

Finding the Range

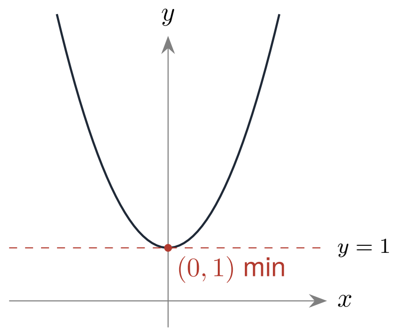
Sketch the graph; the range is the set of y-values it covers.

♦ Example 1.

Find the range of $f(x) = x^2 + 1$, $x \in \mathbb{R}$.

Solution.

Sketch $y = x^2 + 1$.



The graph starts at $y = 1$ (its lowest point) and increases without bound. Hence the range is

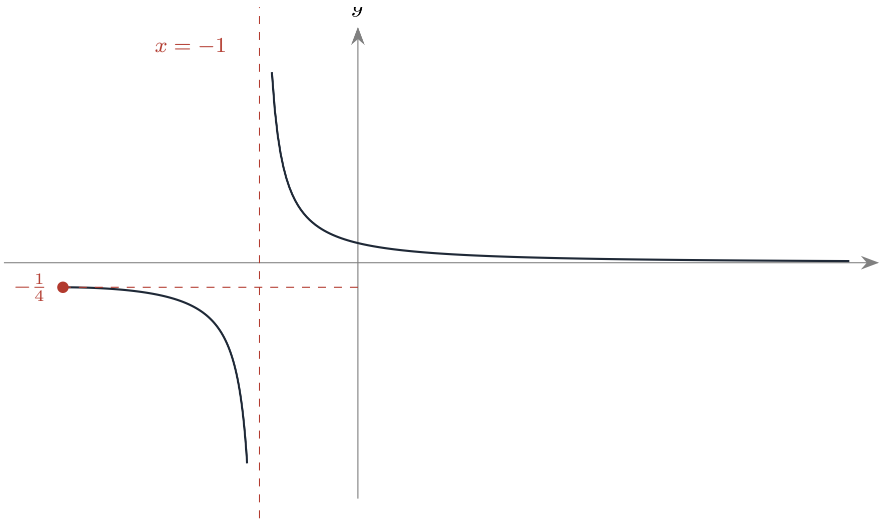
$$\mathsf{R}_f \;=\; [1, \infty).$$

♦ Example 2.

Find the range of $f(x) = \dfrac{1}{x^2 + 6x + 5}$, $x \geq -3$.

Solution.

Using the GC, sketch $y = f(x)$ for $x \geq -3$ (two branches, separated by the asymptote $x = -1$):



Read off each branch as a domain $\to$ range mapping:

- Left branch: domain $= [-3, -1)$, hence range $= \left(-\infty, -\frac{1}{4}\right]$.
- Right branch: domain $= (-1, \infty)$, hence range $= (0, \infty)$.

$$\mathrm{R}_f \;=\; \left(-\infty, -\tfrac{1}{4}\right] \cup (0, \infty).$$

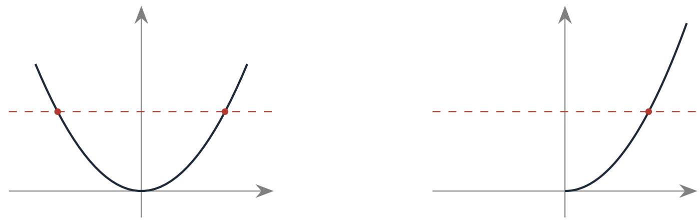
ONE-TO-ONE FUNCTIONS

A function f is **one-to-one** (1-1) if no two different inputs give the same output.

A strictly increasing or strictly decreasing function is one-to-one: no output is repeated, since the value moves in one direction only.

The Horizontal Line Test

A function is 1-1 if every horizontal line cuts its graph **at most once**.



Not 1-1 (line cuts twice) *1-1 (line cuts once)*

Restriction of Domain

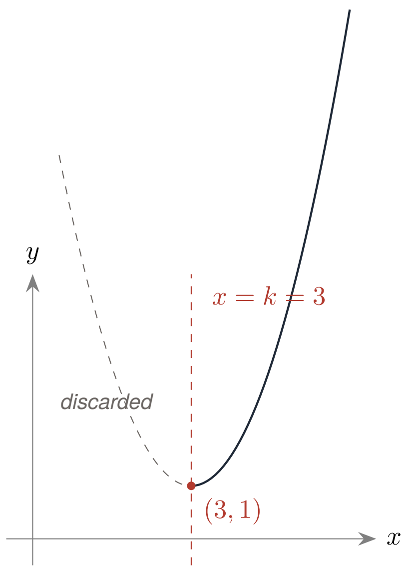
If f is not 1-1 on its full domain (like the parabola above), we can restrict the domain to a sub-interval on which f is 1-1.

Example 3.

The function f is defined by $f(x) = (x-3)^2 + 1$, $x \in \mathbb{R}$. Find the smallest value of k such that f is 1-1 on $x \geq k$.

Solution.

The smallest k for which f is 1-1 on $[k, \infty)$ is $k = 3$.

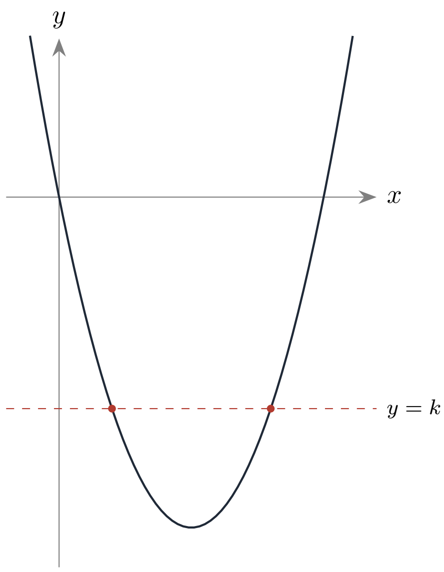


♦ Example 4.

The function f is defined by $f(x) = x^2 - 5x$, $x \in \mathbb{R}$. Show that f is not one-to-one, and explain why f^{-1} does not exist.

Solution.

Draw a horizontal line $y = k$ across the graph of f:



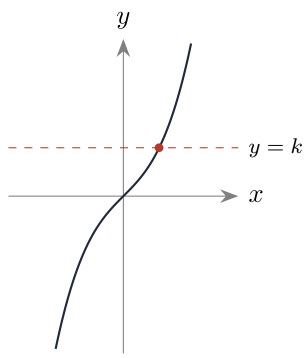
Since the line $y = k$ ($k \in \mathbb{R}$) cuts the graph more than once, f is not one-to-one. Hence f^{-1} does not exist.

♦ Example 5.

The function f is defined by $f(x) = x^3 + x$, $x \in \mathbb{R}$. Show that f is one-to-one.

Solution.

Sketch $y = f(x)$:



Every horizontal line $y = k$ ($k \in \mathbb{R}$) cuts the graph at most once. Hence f is one-to-one.

INVERSE FUNCTIONS

The **inverse** of f, written f^{-1}, is the function that undoes f:

$$f^{-1}\big(f(x)\big) = x \qquad \text{and} \qquad f\big(f^{-1}(y)\big) = y.$$

Existence of f^{-1}
f^{-1} exists if and only if f is 1-1 (passes the horizontal line test).

Domain and Range of the Inverse

Since f^{-1} reverses the roles of input and output:

$$\mathsf{D}_{f^{-1}} = \mathsf{R}_f, \qquad \mathsf{R}_{f^{-1}} = \mathsf{D}_f.$$

Finding the Rule of f^{-1}

♦ **METHOD (find the rule of f^{-1})**
1. Check f is 1-1, so f^{-1} exists.
2. Write $y = f(x)$.
3. Solve for x in terms of y. If a square root appears, choose the sign that matches the original domain.
4. Swap labels: $x \leftrightarrow y$.
5. The new rule is $f^{-1}(x)$. State its domain: it equals R_f.

♦ Example 6.

The function f is defined by $f(x) = \dfrac{1}{x^2+6x+5}$, $x \geq -3$. Find f^{-1} and state its domain.

Solution.

From Example 2, on the domain $x \geq -3$ the function is 1-1 (the two branches $[-3,-1)$ and $(-1,\infty)$ produce disjoint y-values), so f^{-1} exists.

Step 1: Write $y = f(x)$ and rearrange.

$$y \;=\; \frac{1}{x^2+6x+5}.$$

Complete the square on the denominator: $x^2+6x+5 = (x+3)^2 - 4$. Hence

$$y\left[(x+3)^2 - 4\right] = 1.$$

Step 2: Solve for x.

$$(x+3)^2 \;=\; \tfrac{1}{y} + 4 \quad\Longrightarrow\quad x+3 \;=\; \pm\sqrt{\tfrac{1}{y}+4}.$$

Step 3: Choose the right sign. The original domain is $x \geq -3$, so $x+3 \geq 0$. Take the positive root:

$$x \;=\; -3 + \sqrt{\tfrac{1}{y}+4}.$$

Step 4: Swap labels and state domain.

$$f^{-1}(x) \;=\; -3 + \sqrt{\tfrac{1}{x}+4}, \qquad x \in \mathsf{D}_{f^{-1}} = \mathsf{R}_f = \left(-\infty, -\tfrac{1}{4}\right] \cup (0,\infty).$$

♦ Example 7.

The function f is defined by $f(x) = (x-a)^2 + 2a$, $x \geq a$, where a is a real constant. Find f^{-1}, giving the rule and its domain in terms of a.

Solution.

Step 1: Confirm f^{-1} exists. On $x \geq a$ the graph is the right half of a parabola with vertex at $x = a$, so f is increasing and hence 1-1. Thus f^{-1} exists.

Step 2: Write $y = f(x)$ and solve for x.

$$y = (x-a)^2 + 2a \quad \Longrightarrow \quad (x-a)^2 = y - 2a \quad \Longrightarrow \quad x - a = \pm\sqrt{y-2a}.$$

Step 3: Choose the sign from the domain. The domain is $x \geq a$, so $x - a \geq 0$. Take the positive root:

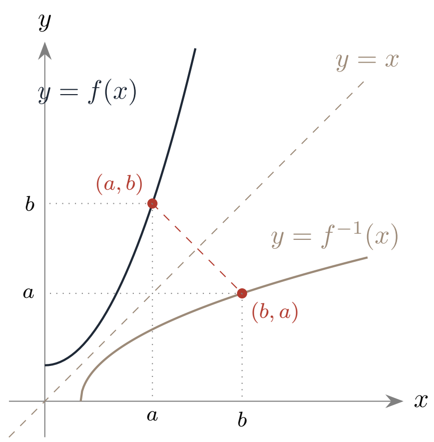
$$x = a + \sqrt{y - 2a}.$$

Step 4: Swap labels and state the domain. The least output is $f(a) = 2a$, so $\mathsf{R}_f = [2a, \infty)$. Hence

$$f^{-1}(x) = a + \sqrt{x - 2a}, \qquad x \in \mathsf{D}_{f^{-1}} = \mathsf{R}_f = [2a, \infty).$$

♦ **Note.** Carry the parameter a through every line; do NOT substitute a number for it. The sign in Step 3 is still decided by the original domain, exactly as in the numerical case.

The Graph of f^{-1}



Hence the coordinates swap: if (a, b) lies on $y = f(x)$, then (b, a) lies on $y = f^{-1}(x)$. This is exactly the effect of reflection in $y = x$.

♦ Example 8.

The function f is defined by $f(x) = \frac{1}{2}\sqrt{36 - (x-3)^2}$, $3 \leq x < 9$.

(a) Find f^{-1}.
(b) On a single diagram, sketch $y = f(x)$, $y = f^{-1}(x)$, and $y = ff^{-1}(x)$.

Solution.

(a) Let $y = \frac{1}{2}\sqrt{36 - (x-3)^2}$.

Multiply both sides by 2, then square (we know $y \geq 0$, so squaring is safe):

$$2y = \sqrt{36 - (x-3)^2} \quad \Longrightarrow \quad 4y^2 = 36 - (x-3)^2.$$

$$(x-3)^2 = 36 - 4y^2 \quad \Longrightarrow \quad x - 3 = \pm\sqrt{36 - 4y^2}.$$

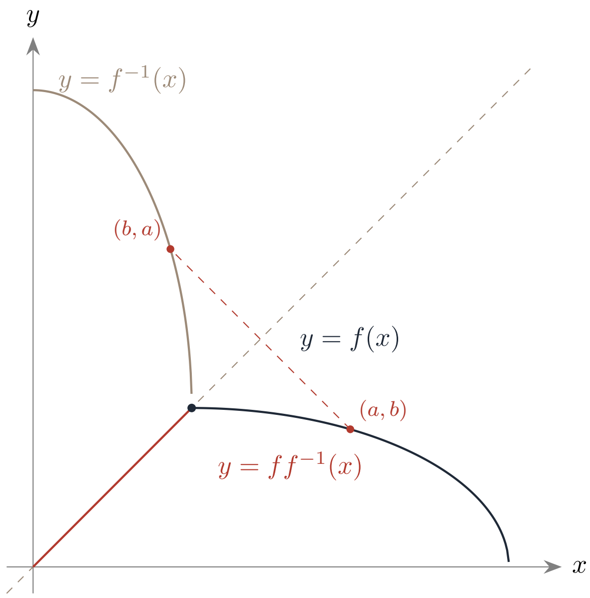
Since the original domain has $x \geq 3$, $x - 3 \geq 0$, so take the $+$ sign:

$$x = 3 + 2\sqrt{9 - y^2}.$$

Swap labels:

$$f^{-1}(x) = 3 + 2\sqrt{9 - x^2}, \qquad x \in \mathsf{D}_{f^{-1}} = \mathsf{R}_f = (0, 3].$$

(b) The composite $y = ff^{-1}(x)$ simplifies to $y = x$.



Intersection of $y = f(x)$ and $y = f^{-1}(x)$

♦ **Example 9.**

The function f is defined by $f(x) = x^2 - 3$, $x \geq 0$. Find the exact x-coordinate where $y = f(x)$ meets $y = f^{-1}(x)$.

Solution.

♦ **Note.** $y = f(x)$ and $y = f^{-1}(x)$ meet on the line $y = x$, so any intersection satisfies $f(x) = x$.

Solve $f(x) = x$:

$$x^2 - 3 = x \implies x^2 - x - 3 = 0.$$

$$x = \frac{1 \pm \sqrt{1 + 12}}{2} = \frac{1 \pm \sqrt{13}}{2}.$$

Since $x \geq 0$, take the positive root: $x = \dfrac{1 + \sqrt{13}}{2}$.

Self-Inverse Functions

A function is **self-inverse** if $f^{-1} = f$, equivalently $f(f(x)) = x$ for all x in the domain.

Example 10.

Show that $f(x) = \dfrac{x+2}{x-1}$, $x \in \mathbb{R}, x \neq 1$, is self-inverse.

Solution.

Find f^{-1}. Let $y = \dfrac{x+2}{x-1}$. Then

$$y(x-1) = x+2 \quad\Longrightarrow\quad yx - y = x + 2 \quad\Longrightarrow\quad x(y-1) = y+2.$$

$$x = \frac{y+2}{y-1}.$$

Swap labels:

$$f^{-1}(x) = \frac{x+2}{x-1} = f(x), \qquad x \neq 1.$$

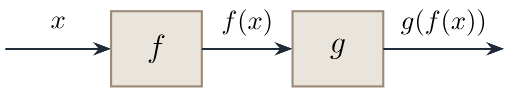
Since $f^{-1} = f$, f is self-inverse. $\square$

COMPOSITE FUNCTIONS

The **composite** gf is the function you get by feeding the output of f into g:

$$(gf)(x) \;=\; g\big(f(x)\big).$$

The composite is two machines in series: the output of f becomes the input of g.



When Does gf Exist?

Existence of the composite gf
gf exists if and only if $R_f \subseteq D_g$.
(Every output of f must be a legal input for g.)

Notation: f^2 versus $[f(x)]^2$

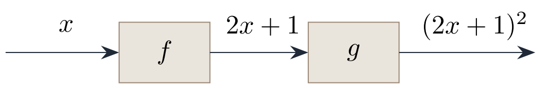
f^2 means $f(f(x))$, applying f twice. It does NOT mean $[f(x)]^2$.

Example 11.

Let $f(x) = 2x + 1$ and $g(x) = x^2$, both with domain $\mathbb{R}$. Find $gf(x)$ and $fg(x)$, and check whether they are equal.

Solution.

Compute gf: feed x through f first, then through g.



$$gf(x) \;=\; g(2x+1) \;=\; (2x+1)^2 \;=\; 4x^2 + 4x + 1.$$

Compute fg: feed x through g first, then through f.

$$fg(x) \;=\; f(x^2) \;=\; 2x^2 + 1.$$

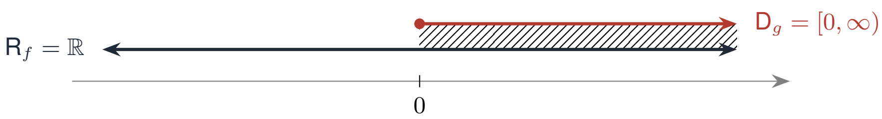
Equal? No: $4x^2 + 4x + 1 \neq 2x^2 + 1$ in general. Composition is not commutative; the ORDER matters.

♦ Example 12.

The functions f and g are defined by $f(x) = x - 3$, $x \in \mathbb{R}$, and $g(x) = \sqrt{x}$, $x \geq 0$. Determine whether the composite gf exists.

Solution.

Recall that gf exists if and only if $\mathsf{R}_f \subseteq \mathsf{D}_g$. Here $\mathsf{R}_f = \mathbb{R}$ and $\mathsf{D}_g = [0, \infty)$. Plot them on a number line:



Hence $\mathsf{R}_f \not\subseteq \mathsf{D}_g$, and gf does NOT exist.

♦ Example 13.

The functions f and g are defined by $f(x) = \sqrt{x - 4}$, $x \geq 4$, and $g(x) = x^2$, $x \in \mathbb{R}$. Determine whether fg exists.

Solution.

For fg the inner function is g and the outer is f, so the test is $\mathsf{R}_g \subseteq \mathsf{D}_f$.

Here $\mathsf{R}_g = [0, \infty)$ and $\mathsf{D}_f = [4, \infty)$. Plot them on a number line:

Hence $\mathsf{R}_g \not\subseteq \mathsf{D}_f$, and fg does NOT exist.

Finding the Range of gf

Trace the chain of domains and ranges:

$$\mathsf{D}_f \xrightarrow{\;f\;} \mathsf{R}_f \xrightarrow{\;g\;} \mathsf{R}_{gf}$$

♦ Example 14.

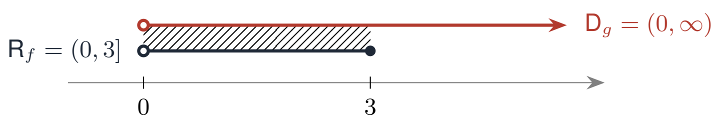
The functions f and g are defined by

$$f(x) = \tfrac{1}{2}\sqrt{36-(x-3)^2}, \quad 3 \le x < 9, \qquad g(x) = -\tfrac{1}{8}x^2(x-5), \quad x > 0.$$

Show that gf exists and find R_{gf}.

Solution.

Step 1: Existence check. Here $\mathrm{R}_f = (0,3]$ and $\mathrm{D}_g = (0,\infty)$. Plot them on a number line:



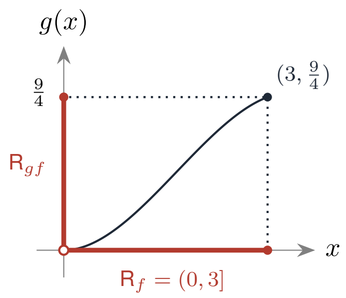
Since $\mathrm{R}_f \subseteq \mathrm{D}_g$, gf exists.

Step 2: Find the range of gf. The outputs of gf are the values g produces from inputs in $(0,3]$. Hence we determine the values g takes on the interval $(0,3]$.

Sketch $g(x) = -\frac{1}{8}x^2(x-5)$ on $(0,3]$. At the endpoints:

$$g(0) = 0, \qquad g(3) = -\tfrac{1}{8}(9)(-2) = \tfrac{9}{4}.$$

Between 0 and 3, g is positive (since $x > 0$ and $x-5 < 0$ give $-x^2(x-5) > 0$) and increases smoothly:



Reading off: g takes values from 0 (not inclusive, since $x = 0$ is not in D_g) up to and including $\frac{9}{4}$.

$$\mathrm{R}_{gf} \;=\; \left(0,\, \tfrac{9}{4}\right].$$

♦ Example 15.

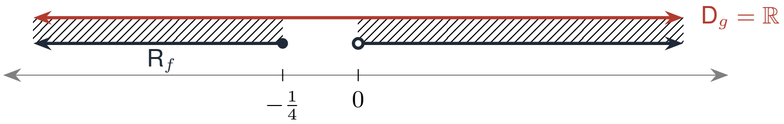
The functions f and g are defined by

$$f(x) = \frac{1}{x^2+6x+5}, \quad x \geq -3, \qquad g(x) = e^x, \quad x \in \mathbb{R}.$$

Show that gf exists and find R_{gf}. *(Use* $\mathsf{R}_f = (-\infty, -\frac{1}{4}] \cup (0, \infty)$ *from the same function in Example 2.)*

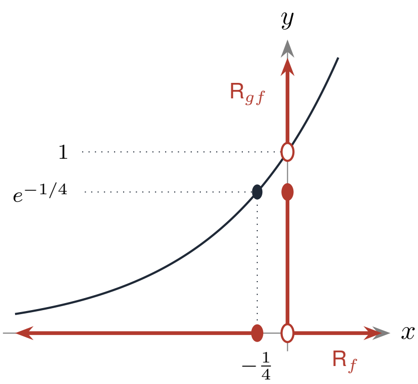
Solution.

Here $\mathsf{R}_f = (-\infty, -\frac{1}{4}] \cup (0, \infty)$ and $\mathsf{D}_g = \mathbb{R}$. Plot them on a number line:



Since $\mathsf{R}_f \subseteq \mathsf{D}_g$, gf exists.

Range. Apply $g(x) = e^x$ to each piece of $\mathsf{R}_f = (-\infty, -\frac{1}{4}] \cup (0, \infty)$. From the graph of $y = e^x$:



On $(-\infty, -\frac{1}{4}]$**:** e^x runs from 0 (not inclusive) up to $e^{-1/4}$, giving $(0,\, e^{-1/4}]$.

On $(0, \infty)$**:** e^x runs from 1 (not inclusive) to ∞, giving $(1, \infty)$.

$$\mathsf{R}_{gf} = \left(0,\, e^{-1/4}\right] \cup (1, \infty).$$

Composition of f with f^{-1}

Applying f then f^{-1}, or f^{-1} then f, returns the original input. The composite is the identity.

♦ Example 16.

Let $f(x) = 2x - 1$, $x \in \mathbb{R}$. Find $f^{-1}(x)$ and check both $f^{-1}f(x) = x$ and $ff^{-1}(x) = x$.

Solution.

Solve $y = 2x - 1$ for x: $x = \frac{y+1}{2}$. Hence

$$f^{-1}(x) \;=\; \tfrac{x+1}{2}, \qquad x \in \mathbb{R}.$$

Check $f^{-1}f(x)$ (do f, then f^{-1}):

$$f^{-1}f(x) \;=\; f^{-1}(2x-1) \;=\; \tfrac{(2x-1)+1}{2} \;=\; \tfrac{2x}{2} \;=\; x. \quad \checkmark$$

Check $ff^{-1}(x)$ (do f^{-1}, then f):

$$ff^{-1}(x) \;=\; f\big(\tfrac{x+1}{2}\big) \;=\; 2 \cdot \tfrac{x+1}{2} - 1 \;=\; (x+1) - 1 \;=\; x. \quad \checkmark$$